Int. J. Solids Structures, 1966, Vol. 2, pp. 157 to 166. Pergamon Press Ltd. Printed in Great Britain

THEORY OF ELASTIC PLATES IN THE REFERENCE STATE

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Abstract—A non-linear plate theory is developed on the basis of three-dimensional theory of elasticity in terms of a reference state. This includes the effects of transverse shear and normal strains, acceleration, material heterogeneity and anisotropy, and a temperature field. The results are valid for plates of constant thickness and contain those of certain earlier non-linear plate theories as special cases.

1. INTRODUCTION

WHEN the state of stress in a deformed body is represented by the stress tensor, s^{ij}, measured per unit area of the undeformed body, a new form of the conventional Cauchy's laws of motion can be obtained. For the non-polar case, it follows that s^{ij} is a symmetric tensor. It is noteworthy that this tensor arises when the stress vector per unit area of the undeformed body, associated with a surface in the deformed body, is referred to base vectors in the deformed body. If the same vector be referred to base vectors in the undeformed body, a corresponding stress tensor, t^{ij}, ensues. The historical development of these aspects of the measure of stress can be traced with the help of references given by Truesdell and Toupin [1a] and Reissner [2]; a summary of the main results has also been presented by Green and Adkins [3].

In what follows, a non-linear theory of elastic plates "in terms of a reference state" will be extracted from three-dimensional continuum dynamics in terms of the stress tensor s^{ij}. The non-linear plate equations of motion are first derived by integrating the version of Cauchy's laws of motion in terms of a reference state across the variable thickness of the undeformed plate. This method—which, for the classical theory of elastic plates, dates back to the last century and has in recent years been employed in developing plate and shell theories that include the effect of transverse shear deformation—has so far been applied to the conventional version of Cauchy's laws in terms of stress per unit area of the deformed body. An exception is due to Koppe [4] who employed an analogous procedure, based on the related equations involving t^{ij}, in obtaining plate equations of equilibrium. The effects of eventual simplifying assumptions, however, become easier to trace, and a consistent system of equations to derive, when a variational procedure is adopted. Accordingly, the fundamental equations of the non-linear theory of elastic plates in the reference state are next derived with the help of a variational principle of three-dimensional elasticity. The resulting equations comprise non-linear strain–displacement relations and constitutive equations as well as mixed boundary conditions in addition to non-linear plate equations of motion. The theory which includes the effects of transverse shear and normal strains, acceleration, material heterogeneity and anisotropy, and a prescribed steady temperature field, is then shown to yield, under Kirchhoff's hypothesis, a classical non-linear plate theory. Lastly, complete linearization of our results leads to the familiar small-deflection theory of plates. The original version of our theory is particularly suitable for use in the non-linear analysis of sandwich [5], inflatable and, indeed, other types of heterogeneous and anisotropic plates, when subjected to mechanical and thermal loads, so long as the effect of transverse shear deformation is known to be appreciable.

2. VARIATIONAL PRINCIPLE

The Hellinger–Reissner principle discussed by Truesdell and Toupin [1b] on the basis of previous work by Hellinger [6] and Reissner [2] leads to Cauchy's first law of motion and the mixed boundary conditions of the theory of elasticity in terms of a reference state within a two point field description. The same has also been illustrated by Doyle and Ericksen [7a]. In order to render the present work self-contained, it is our purpose in this section to reformulate a modified version, after the procedure due to Hu [8] and Washizu [9, 10], of the Hellinger–Reissner principle, in general convected coordinates, θ^i, and using the stress tensor s^{ij}. We shall thus obtain, among the resulting basic equations of non-linear elasticity, the non-linear strain–displacement relations

$$\gamma_{ij} = \tfrac{1}{2}(v_i|_j + v_j|_i + v^r|_i v_r|_j), \tag{1}$$

where γ_{ij} is the strain tensor defined by Green and Adkins [3]; v^i and v_i are the components of the displacement vector with respect to base vectors of the undeformed body, a vertical line denotes covariant differentiation with respect to θ^i using the metric tensor of the undeformed body, and Latin indices take the values 1, 2, 3 and are summed when repeated.

Let ${}_0V$ be the volume of the undeformed body, and ${}_0A_s$ and ${}_0A_v$ be the two parts of its total boundary where the stress and displacement vectors are prescribed, respectively. Let dV and dA denote the corresponding elements of volume and area, respectively. Let s^i_* be the components of the stress vector, per unit area of the undeformed body, referred to base vectors in the undeformed body. In the case when there is a strain energy, let Σ^* denote the strain energy function measured per unit volume of the undeformed body. Let ${}_0f^i$ and ${}_0F^i$ be, respectively, the components of acceleration with respect to the undeformed body, and body force per unit mass of the undeformed body of density ρ_0. Let δ indicate variation, and prescribed quantities be denoted by a tilde. Then, rephrasing the version given by Truesdell and Toupin [1b], the modified Hellinger–Reissner theorem asserts that the variational principle

$$\begin{aligned}\int_{{}_0V} \rho_0({}_0f^i - {}_0F^i)\delta v_i \, \mathrm{d}V = \delta\Big[\int_{{}_0V} (s^{ij}\gamma_{ij} - \Sigma^*)\,\mathrm{d}V - \int_{{}_0V} \tfrac{1}{2}s^{ij}(v_i|_j + v_j|_i + v^r|_i v_r|_j)\,\mathrm{d}V \\ + \int_{{}_0A_s} \tilde{s}^i_* v_i \,\mathrm{d}A + \int_{{}_0A_v} s^i_*(v_i - \tilde{v}_i)\,\mathrm{d}A\Big],\end{aligned} \tag{2}$$

where γ_{ij}, s^{ij}, v_i, and s^i_* are varied independently, is equivalent to Cauchy's first law in ${}_0V$, to the stress boundary condition on the part ${}_0A_s$ of the boundary, to the displacement boundary condition on the remaining part ${}_0A_v$, and to the constitutive equations and strain displacement relations in ${}_0V$, when both the symmetries of γ_{ij} and s^{ij} are used.

To establish this theorem, we carry out the indicated variation in equation (2). Using Green's transformation and combining the resulting volume and surface integrals, we obtain

$$\int_{{}_0V}\left\langle\delta\gamma_{ij}\left[s^{ij}-\frac{1}{2}\left(\frac{\partial\Sigma^*}{\partial\gamma_{ij}}+\frac{\partial\Sigma^*}{\partial\gamma_{ji}}\right)\right]+\delta s^{ij}[\gamma_{ij}+\tfrac{1}{2}(v_i|_j+v_j|_i+v^r|_iv_r|_j)]\right.$$
$$\left.+\delta v_i\{[s^{jr}(\delta^i_r+v^i|_r)]|_j+\rho_0({}_0F^i-{}_0f^i)\}\right\rangle\,\mathrm{d}V+\int_{{}_0A_s}[\tilde{s}^i_*-{}_0n_js^{jr}(\delta^i_r+v^i|_r)]\delta v_i\,\mathrm{d}A \qquad (3)$$
$$+\int_{{}_0A_v}\{[s^i_*-{}_0n_js^{jr}(\delta^i_r+v^i|_r)]\delta v_i+(v_i-\tilde{v}_i)\delta s^i_*\}\,\mathrm{d}A=0,$$

where δ^i_j is the Kronecker symbol, and ${}_0n_i$ stands for the components of the unit normal with respect to base vectors in the undeformed body. For independent and arbitrary variations of the indicated quantities, it follows from equation (3)

—in ${}_0V$

$$s^{ij}=\frac{1}{2}\left(\frac{\partial\Sigma^*}{\partial\gamma_{ij}}+\frac{\partial\Sigma^*}{\partial\gamma_{ji}}\right), \qquad (4)$$

$$\gamma_{ij}=\tfrac{1}{2}(v_i|_j+v_j|_i+v^r|_iv_r|_j), \qquad (5)$$

$$[s^{jr}(\delta^i_r+v^i|_r)]|_j+\rho_{00}F^i=\rho_{00}f^i, \qquad (6)$$

—on ${}_0A_s$

$$\tilde{s}^i_*=s^i_*={}_0n_js^{jr}(\delta^i_r+v^i|_r), \qquad (7)$$

—on ${}_0A_v$

$$\tilde{v}_i=v_i. \qquad (8)$$

Equations (4)–(8) constitute a known version of the fundamental equations of the non-linear theory of elasticity in terms of a reference state, as given, for instance, by Green and Adkins [3], and thus verify the theorem. Similar equations have also been discussed, among others, by Novozhilov [11], Landau and Lifshitz [12], Pearson [13], Prager [14] and Eringen [15].

The theorem given by Reissner [2] now follows from equation (2) by using the inverse

$$\gamma_{ij}=\frac{1}{2}\left(\frac{\partial W^*}{\partial s^{ij}}+\frac{\partial W^*}{\partial s^{ji}}\right), \qquad \Sigma^*=s^{ij}\gamma_{ij}-W^* \qquad (9)$$

of the Legendre transformation

$$s^{ij}=\frac{1}{2}\left(\frac{\partial\Sigma^*}{\partial\gamma_{ij}}+\frac{\partial\Sigma^*}{\partial\gamma_{ji}}\right), \qquad W^*=s^{ij}\gamma_{ij}-\Sigma^*, \qquad (10)$$

provided the Hessian of Σ^* does not vanish and the strain–displacement relations (5) are imposed *a priori*. Thus, W^* is the complementary energy function per unit volume of the undeformed body. For further contributions on this matter, we refer the reader to Manacorda [16] who presented still another formulation, reproduced by Doyle and Ericksen [7b], and to Koppe [17] who considered the derivation of related variational principles of non-linear elasticity.

Clearly, to the various stages of partial linearization in the strain–displacement relations correspond simplified versions of equation (2) and, hence, of equations (6) and (7). A recent reformulation of equation (2), in the form of a generalized Hamilton's principle and in terms of the elongation and mean rotation tensors, has been given by Yu [18] who also evaluated thus simplified non-linear theories of elasticity earlier available in the literature.

For later use, we record the following linear version of equation (4)

$$s^{ij} = C^{ijrs}(\gamma_{rs} - \alpha_{rs}\Theta), \tag{11}$$

where, for heterogeneous and anisotropic material in the presence of a prescribed steady temperature field, $\Theta(\theta^i)$, C^{ijrs} are the isothermal stiffnesses and α_{ij} are strain–temperature coefficients at constant stress. The following symmetry relations

$$\left.\begin{aligned} C^{ijrs} &= C^{jirs} = C^{ijsr} = C^{rsij}, \\ \alpha_{ij} &= \alpha_{ji} \end{aligned}\right\} \tag{12}$$

are satisfied. For a medium having elastic symmetry with respect to the surface $\theta^3 = \text{const}$, following Green and Zerna [19], equation (11) reduces to

$$\left.\begin{aligned} s^{\alpha\beta} &= C^{\alpha\beta\delta\lambda}(\gamma_{\delta\lambda} - \alpha_{\delta\lambda}\Theta) + C^{\alpha\beta 33}(\gamma_{33} - \alpha_{33}\Theta), \\ s^{\alpha 3} &= 2C^{\alpha 3\beta 3}(\gamma_{\beta 3} - \alpha_{\beta 3}\Theta), \\ s^{33} &= C^{33\alpha\beta}(\gamma_{\alpha\beta} - \alpha_{\alpha\beta}\Theta) + C^{3333}(\gamma_{33} - \alpha_{33}\Theta), \end{aligned}\right\} \tag{13}$$

where Greek indices take the values 1, 2 and are summed when repeated. Within the conventional representation of stress, relations such as equation (13), but with $\Theta = 0$, have been employed by Naghdi [20] for elastic anisotropic shells; expressions equivalent to equation (11) are given, for instance, by Hearmon [21] and Nowacki [22].

3. NON-LINEAR PLATE THEORY

(*a*) *Preliminaries*

When referring to the plate, the original set of general convected coordinates will be identified with a set of geodesic normal [23] convected coordinates—$\theta^3 = 0$ being the reference plane—so that the components of the metric tensor of the undeformed plate space are given by

$$g_{\alpha\beta} = a_{\alpha\beta}, \qquad g_{\alpha 3} = 0, \qquad g_{33} = 1, \tag{14}$$

where $a_{\alpha\beta}$ is the metric tensor of the reference plane of the undeformed plate. The undeformed plate of variable thickness $|h_2 - h_1|$ is then defined as the region bounded by the two plane faces, $\theta^3 = h_1(\theta^\alpha)$ and $\theta^3 = h_2(\theta^\alpha)$, and the edge boundary which is taken as a cylindrical surface intersecting the reference plane, ${}_0a$, along a simple closed curve ${}_0c$, and whose generators lie along the normal to the reference plane. A simply connected plate will be assumed; no singularities of any kind are supposed to be present.

In order to illustrate our two methods of deriving non-linear plate equations of motion in terms of a reference state, the displacement components will be taken as

$$v_i = u_i(\theta^\alpha, \tau) + \theta^3\psi_i(\theta^\alpha, \tau), \tag{15}$$

where τ denotes time. From equations (5) and (15)

$$\left.\begin{aligned} \gamma_{\alpha\beta} &= {}_0\gamma_{\alpha\beta} + \theta^3\,{}_1\gamma_{\alpha\beta} + (\theta^3)^2\,{}_2\gamma_{\alpha\beta}, \\ \gamma_{\alpha 3} &= {}_0\gamma_{\alpha 3} + \theta^3\,{}_1\gamma_{\alpha 3}, \\ \gamma_{33} &= {}_0\gamma_{33}, \end{aligned}\right\} \tag{16}$$

where ${}_k\gamma_{\alpha\beta}$ $(k = 0, 1, 2)$, ${}_l\gamma_{\alpha 3}$ $(l = 0, 1)$ and ${}_m\gamma_{33}$ $(m = 0)$ are known functions of u_i and ψ_i.

For later use, we introduce the following definitions for the stress and couple resultants per unit length of coordinate curves on ${}_0a$, and effective external loads per unit area of ${}_0a$.

$$\left.\begin{aligned}
&\begin{Bmatrix} N^{\alpha\beta} = N^{\beta\alpha} \\ M^{\alpha\beta} = M^{\beta\alpha} \\ K^{\alpha\beta} = K^{\beta\alpha} \end{Bmatrix} = \int_{h_1}^{h_2} s^{\alpha\beta} \begin{Bmatrix} 1 \\ \theta^3 \\ (\theta^3)^2 \end{Bmatrix} \mathrm{d}\theta^3, \\
&\begin{Bmatrix} Q^\alpha \\ T^\alpha \end{Bmatrix} = \int_{h_1}^{h_2} s^{\alpha 3} \begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} \mathrm{d}\theta^3, \qquad N^{33} = \int_{h_1}^{h_2} s^{33}\, \mathrm{d}\theta^3, \\
&\begin{Bmatrix} \mathscr{F}^i \\ \mathscr{M}^i \end{Bmatrix} = \int_{h_1}^{h_2} \rho_{00} F^i \begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} \mathrm{d}\theta^3, \qquad \begin{Bmatrix} \mathscr{f}^i \\ \mathscr{m}^i \end{Bmatrix} = \int_{h_1}^{h_2} \rho_{00} f^i \begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} \mathrm{d}\theta^3, \\
&\begin{Bmatrix} p^i \\ c^i \end{Bmatrix} = \begin{Bmatrix} \mathscr{F}^i \\ \mathscr{M}^i \end{Bmatrix} + \begin{Bmatrix} p_2^i - p_1^i \\ c_2^i - c_1^i \end{Bmatrix}, \\
&\begin{Bmatrix} p_n^\alpha \\ c_n^\alpha \end{Bmatrix} = \left[\begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} \{ s^{3\beta}(\delta_\beta^\alpha + u^\alpha|_\beta + \theta^3 \psi^\alpha|_\beta) + s^{33}\psi^\alpha \} \right]_{\theta^3 = h_n} \\
&\qquad (n = 1, 2), \\
&\begin{Bmatrix} p_n^3 \\ c_n^3 \end{Bmatrix} = \left[\begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} \{ s^{3\alpha}(u_{3,\alpha} + \theta^3 \psi_{3,\alpha}) + s^{33}(1 + \psi_3) \} \right]_{\theta^3 = h_n} \\
&\qquad (n = 1, 2),
\end{aligned}\right\} \quad (17)$$

where a comma denotes partial differentiation with respect to the indicated variable.

(b) Non-linear plate equations of motion

Prior to integration with respect to θ^3, the three-dimensional equations of motion (6) are put into the following form

$$[s^{\alpha\beta}(\delta_\beta^\delta + u^\delta|_\beta + \theta^3\psi^\delta|_\beta)]|_\alpha + (s^{\alpha 3}\psi^\alpha)|_\alpha + [s^{\beta 3}(\delta_\beta^\delta + u^\delta|_\beta + \theta^3\psi^\delta|_\beta) + s^{33}\psi^\delta]_{,3} + \rho_{00}F^\delta = \rho_{00}f^\delta, \quad (18)$$

$$[s^{\alpha\beta}(u_{3,\beta} + \theta^3\psi_{3,\beta})]|_\alpha + [s^{\alpha 3}(1 + \psi_3)]|_\alpha + [s^{\alpha 3}(u_{3,\alpha} + \theta^3\psi_{3,\alpha}) + s^{33}(1 + \psi_3)]_{,3} + \rho_{00}F^3 = \rho_{00}f^3, \quad (19)$$

$$\begin{aligned}
&[\theta^3 s^{\alpha\beta}(\delta_\beta^\delta + u^\delta|_\beta + \theta^3\psi^\delta|_\beta)]|_\alpha + (\theta^3 s^{\alpha 3}\psi^\delta)|_\alpha - s^{\alpha 3}(\delta_\alpha^\delta + u^\delta|_\alpha + \theta^3\psi^\delta|_\alpha) - s^{33}\psi^\delta \\
&\qquad + \{\theta^3[s^{\alpha 3}(\delta_\alpha^\delta + u^\delta|_\alpha + \theta^3\psi^\delta|_\alpha) + s^{33}\psi^\delta]\}_{,3} + \theta^3\rho_{00}F^\delta = \theta^3\rho_{00}f^\delta, \quad (20)
\end{aligned}$$

$$\begin{aligned}
&[\theta^3 s^{\alpha\beta}(u_{3,\beta} + \theta^3\psi_{3,\beta})]|_\alpha + [\theta^3 s^{\alpha 3}(1 + \psi_3)]|_\alpha - s^{\alpha 3}(u_{3,\alpha} + \theta^3\psi_{3,\alpha}) - s^{33}(1 + \psi_3) \\
&\qquad + \{\theta^3[s^{\alpha 3}(u_{3,\alpha} + \theta^3\psi_{3,\alpha}) + s^{33}(1 + \psi_3)]\}_{,3} + \theta^3\rho_{00}F^3 = \theta^3\rho_{00}f^3, \quad (21)
\end{aligned}$$

where equation (15) has been used and equations (20) and (21) follow respectively from (18) and (19) upon multiplication by θ^3. Integrating equations (18)-(21) through the variable thickness of the undeformed plate, we obtain, using the definitions introduced in equation (17), the following system of non-linear equations of motion for a plate in terms of a reference state.

$$[N^{\alpha\beta}(\delta^\delta_\beta+u^\delta|_\beta)]\|_\alpha+(M^{\alpha\beta}\psi^\delta|_\beta)|_\alpha+(Q^\alpha\psi^\delta)|_\alpha+p^\delta = f^\delta, \tag{22}$$

$$(N^{\alpha\beta}u_{3,\beta})|_\alpha+(M^{\alpha\beta}\psi_{3,\beta})|_\alpha+[Q^\alpha(1+\psi_3)]\|_\alpha+p^3 = f^3, \tag{23}$$

$$[M^{\alpha\beta}(\delta^\delta_\beta+u^\delta|_\beta)]\|_\alpha-Q^\alpha(\delta^\delta_\alpha+u^\delta|_\alpha)+(K^{\alpha\beta}\psi^\delta|_\beta)|_\alpha+(T^\alpha|_\alpha-N^{33})\psi^\delta+c^\delta = m^\delta, \tag{24}$$

$$(M^{\alpha\beta}u_{3,\beta})|_\alpha-Q^\alpha u_{3,\alpha}+(K^{\alpha\beta}\psi_{3,\beta})|_\alpha+(T^\alpha|_\alpha-N^{33})(1+\psi_3)+c^3 = m^3. \tag{25}$$

(c) Variational derivation of fundamental equations

In this subsection, we shall use the variational principle established in Section 2 in order to obtain non-linear strain–displacement relations and constitutive equations as well as mixed boundary conditions for an elastic plate of variable thickness and the non-linear equations of motion (22)–(25) which will not be repeated below. We note that the use of the variational principle of virtual work in deriving non-linear theories of shells has been illustrated by Washizu [24].

Performing the integration with respect to θ^3 in equation (3), the fundamental plate equations in terms of a reference state follow, for arbitrary and independent variations of the indicated quantities, through the use of the various relations recorded so far. The results can be summarized as follows.

Strain–displacement relations

$$\left.\begin{aligned}
{}_0\gamma_{\alpha\beta} &= \tfrac{1}{2}(u_\alpha|_\beta+u_\beta|_\alpha+u_\delta|_\alpha u^\delta|_\beta+u_{3,\alpha}u_{3,\beta}),\\
{}_1\gamma_{\alpha\beta} &= \tfrac{1}{2}(\psi_\alpha|_\beta+\psi_\beta|_\alpha+u_\delta|_\alpha\psi^\delta|_\beta+\psi^\delta|_\alpha u_\delta|_\beta+u_{3,\alpha}\psi_{3,\beta}+u_{3,\beta}\psi_{3,\alpha}),\\
{}_2\gamma_{\alpha\beta} &= \tfrac{1}{2}(\psi_\delta|_\alpha\psi^\delta|_\beta+\psi_{3,\alpha}\psi_{3,\beta}),\\
{}_0\gamma_{\alpha 3} &= \tfrac{1}{2}(\psi_\alpha+u_{3,\alpha}+\psi_\beta u^\beta|_\alpha+\psi_3 u_{3,\alpha}),\\
{}_1\gamma_{\alpha 3} &= \tfrac{1}{2}(\psi_{3,\alpha}+\psi_\beta\psi^\beta|_\alpha+\psi_3\psi_{3,\alpha}),\\
{}_0\gamma_{33} &= \tfrac{1}{2}[2\psi_3+\psi_\alpha\psi^\alpha+(\psi_3)^2].
\end{aligned}\right\} \tag{26}$$

Constitutive equations

$$\left.\begin{aligned}
N^{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_0\gamma_{\alpha\beta}}+\frac{\partial\Sigma}{\partial_0\gamma_{\beta\alpha}}\right), & M^{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_1\gamma_{\alpha\beta}}+\frac{\partial\Sigma}{\partial_1\gamma_{\beta\alpha}}\right),\\
K^{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_2\gamma_{\alpha\beta}}+\frac{\partial\Sigma}{\partial_2\gamma_{\beta\alpha}}\right), & Q^{\alpha} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_0\gamma_{\alpha3}}+\frac{\partial\Sigma}{\partial_0\gamma_{3\alpha}}\right),\\
T^{\alpha} &= \frac{1}{2}\left(\frac{\partial\Sigma}{\partial_1\gamma_{\alpha3}}+\frac{\partial\Sigma}{\partial_1\gamma_{3\alpha}}\right), & N^{33} &= \frac{\partial\Sigma}{\partial_0\gamma_{33}},
\end{aligned}\right\} \tag{27}$$

where

$$\Sigma = \int_{h_1}^{h_2}\Sigma^*\,d\theta^3 \tag{28}$$

is the strain energy function per unit area of the undeformed reference plane. In order to obtain linear constitutive equations for a heterogeneous, anisotropic elastic plate subjected to a prescribed steady temperature field, we take, by generalization from Boley and Weiner [25],

$$\Sigma^* = \tfrac{1}{2}s^{ij}(\gamma_{ij}-\alpha_{ij}\Theta), \tag{29}$$

where s^{ij} is given by equation (13). The corresponding constitutive equations are

$$\left.\begin{aligned} N^{\alpha\beta} &= {}_0B^{\alpha\beta\delta\lambda}\,{}_0\gamma_{\delta\lambda}+{}_1B^{\alpha\beta\delta\lambda}\,{}_1\gamma_{\delta\lambda}+{}_2B^{\alpha\beta\delta\lambda}\,{}_2\gamma_{\delta\lambda}+{}_0B^{\alpha\beta33}\,{}_0\gamma_{33}-{}_0\Theta^{\alpha\beta},\\ M^{\alpha\beta} &= {}_1B^{\alpha\beta\delta\lambda}\,{}_0\gamma_{\delta\lambda}+{}_2B^{\alpha\beta\delta\lambda}\,{}_1\gamma_{\delta\lambda}+{}_3B^{\alpha\beta\delta\lambda}\,{}_2\gamma_{\delta\lambda}+{}_1B^{\alpha\beta33}\,{}_0\gamma_{33}-{}_1\Theta^{\alpha\beta},\\ K^{\alpha\beta} &= {}_2B^{\alpha\beta\delta\lambda}\,{}_0\gamma_{\delta\lambda}+{}_3B^{\alpha\beta\delta\lambda}\,{}_1\gamma_{\delta\lambda}+{}_4B^{\alpha\beta\delta\lambda}\,{}_2\gamma_{\delta\lambda}+{}_2B^{\alpha\beta33}\,{}_0\gamma_{33}-{}_2\Theta^{\alpha\beta}\\ Q^{\alpha} &= 2({}_0B^{\alpha3\beta3}\,{}_0\gamma_{\beta3}+{}_1B^{\alpha3\beta3}\,{}_1\gamma_{\beta3}-{}_0\Theta^{\alpha3}),\\ T^{\alpha} &= 2({}_1B^{\alpha3\beta3}\,{}_0\gamma_{\beta3}+{}_2B^{\alpha3\beta3}\,{}_1\gamma_{\beta3}-{}_1\Theta^{\alpha3}),\\ N^{33} &= {}_0B^{\alpha\beta33}\,{}_0\gamma_{\alpha\beta}+{}_1B^{\alpha\beta33}\,{}_1\gamma_{\alpha\beta}+{}_2B^{\alpha\beta33}\,{}_2\gamma_{\alpha\beta}+B^{3333}\,{}_0\gamma_{33}-\Theta^{33}, \end{aligned}\right\}\quad(30)$$

where we have defined

$$\left.\begin{aligned} {}_nB^{\alpha\beta\delta\lambda} &= \int_{h_1}^{h_2} C^{\alpha\beta\delta\lambda}(\theta^3)^n\,\mathrm{d}\theta^3 \qquad (n=0,1,\ldots,4),\\ {}_nB^{\alpha\beta33} &= \int_{h_1}^{h_2} C^{\alpha\beta33}(\theta^3)^n\,\mathrm{d}\theta^3 \qquad (n=0,1,2),\\ {}_nB^{\alpha3\beta3} &= \int_{h_1}^{h_2} C^{\alpha3\beta3}(\theta^3)^n\,\mathrm{d}\theta^3 \qquad (n=0,1,2),\\ B^{3333} &= \int_{h_1}^{h_2} C^{3333}\,\mathrm{d}\theta^3, \end{aligned}\right\}\quad(31)$$

as isothermal plate stiffnesses, and, introduced

$$\left.\begin{aligned} {}_n\Theta^{\alpha\beta} &= \int_{h_1}^{h_2} \Theta(C^{\alpha\beta\delta\lambda}\alpha_{\delta\lambda}+C^{\alpha\beta33}\alpha_{33})(\theta^3)^n\,\mathrm{d}\theta^3 \qquad (n=0,1,2),\\ {}_n\Theta^{\alpha3} &= \int_{h_1}^{h_2} \Theta C^{\alpha3\beta3}\alpha_{\beta3}(\theta^3)^n\,\mathrm{d}\theta^3 \qquad (n=0,1),\\ \Theta^{33} &= \int_{h_1}^{h_2} \Theta(C^{3333}\alpha_{33}+C^{\alpha\beta33}\alpha_{\alpha\beta})\,\mathrm{d}\theta^3, \end{aligned}\right\}\quad(32)$$

as thermal stress and couple resultants per unit length of coordinate curves on ${}_0a$. On account of equation (12), obvious symmetry relations hold for the quantities defined in (31)–(32).

Mixed boundary conditions. Some of the terms in the surface integrals in equation (3) are evaluated as follows. For that part of the boundary where the stress vector is prescribed, i.e. the faces of the plate and part of the edge,

$$\int_{{}_0A_s} \tilde{s}^i_*\,\delta v_i\,\mathrm{d}A = \int_{{}_0c_s} (\tilde{s}^i\delta u_i+\tilde{t}^i\delta\psi_i)\,\mathrm{d}s+\int_{{}_0a_2} (\tilde{p}^i_2\delta u_i+\tilde{c}^i_2\delta\psi_i)\,\mathrm{d}A+\int_{{}_0a_1} (\tilde{p}^i_1\delta u_i+\tilde{c}^i_1\delta\psi_i)\,\mathrm{d}A, \quad (33)$$

where

$$\begin{Bmatrix}\tilde{s}^i\\ \tilde{t}^i\end{Bmatrix} = \int_{h_1}^{h_2} \tilde{s}^i_* \begin{Bmatrix}1\\ \theta^3\end{Bmatrix} d\theta^3, \quad (34)$$

$\mathrm{d}s$ is an element of arc length along ${}_0c$, ${}_0c_s$ is that part of ${}_0c$ where the stress vector is prescribed, and ${}_0a_n$ $(n = 1, 2)$ denotes the faces of the plate. Furthermore, assuming the part where the displacement vector is prescribed to be a portion of the edge of the plate only,

$$\int_{{}_0A_v} \delta s^i_*(v_i-\tilde{v}_i)\,\mathrm{d}A = \int_{{}_0c_v} [\delta s^i(u_i-\tilde{u}_i)+\delta t^i(\psi_i-\tilde{\psi}_i)]\,\mathrm{d}s, \quad (35)$$

where

$$\begin{Bmatrix} s^i \\ t^i \end{Bmatrix} = \int_{h_1}^{h_2} s_*^i \begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} d\theta^3, \tag{36}$$

and ${}_0c_v$ is that part of ${}_0c$ where the displacement vector is prescribed. The resulting stress and displacement boundary conditions are respectively,

—along ${}_0c_s$

$$\left.\begin{aligned} \tilde{s}^\alpha &= s^\alpha = {}_0n_\beta[N^{\beta\delta}(\delta_\delta^\alpha + u^\alpha|_\delta) + M^{\beta\delta}\psi^\alpha|_\delta + Q^\beta\psi^\alpha], \\ \tilde{t}^\alpha &= t^\alpha = {}_0n_\beta[M^{\beta\delta}(\delta_\delta^\alpha + u^\alpha|_\delta) + K^{\beta\delta}\psi^\alpha|_\delta + T^\beta\psi^\alpha], \\ \tilde{s}^3 &= s^3 = {}_0n_\beta[N^{\beta\alpha}u_{3,\alpha} + M^{\beta\alpha}\psi_{3,\alpha} + Q^\beta(1+\psi_3)], \\ \tilde{t}^3 &= t^3 = {}_0n_\beta[M^{\beta\alpha}u_{3,\alpha} + K^{\beta\alpha}\psi_{3,\alpha} + T^\beta(1+\psi_3)], \end{aligned}\right\} \tag{37}$$

and on ${}_0a_n$ $(n = 1, 2)$

$$\tilde{p}^i_n = p^i_n, \qquad \tilde{c}^i_n = c^i_n, \tag{38}$$

—along ${}_0c_v$

$$\tilde{u}_i = u_i, \qquad \tilde{\psi}_i = \psi_i. \tag{39}$$

4. SPECIAL CASES

In this section, we present a few special versions of our results including comparisons with certain theories previously available in the literature. A comprehensive and detailed review of earlier works on the non-linear analysis of elastic plates, with special emphasis on approximate solutions of the basic equations, is beyond the scope of this paper and will be dealt with in a forthcoming memoir. In the meantime, a related survey by Vol'mir [26] should be of considerable interest.

We begin by noting that our results [equations (22)–(25) and (37)] reduce to those of Herrmann and Armenàkas [27], derived by Hamilton's principle, if all terms involving T^α, $K^{\alpha\beta}$, N^{33} and ψ_3 are dropped. The variational principle that we have used has the advantage of yielding non-linear strain–displacement and constitutive equations as well as displacement boundary conditions in addition to the non-linear plate equations of motion and stress boundary conditions.

Taking

$$\left.\begin{aligned} \psi_\alpha &= -u_{3,\beta}(\delta_\alpha^\beta + u^\beta|_\alpha), \\ \psi_3 &= -\tfrac{1}{2}u_{3,\alpha}u_{3,}{}^\alpha, \end{aligned}\right\} \tag{40}$$

we obtain, approximately, the classical case,

$$\gamma_{\alpha 3} = \gamma_{33} \approx 0 \tag{41}$$

of vanishing transverse shear and normal strains. Accordingly, equations (22)–(25) reduce to

$$[N^{\alpha\delta}(\delta_\delta^\beta + u^\beta|_\delta)]|_\alpha - [M^{\alpha\delta}(u_{3,}{}^\beta)|_\delta]|_\alpha - (M^{\alpha\delta}|_\alpha u_{3,}{}^\beta)|_\delta + p^\beta - (c^\delta u_{3,}{}^\beta)|_\delta = \not{f}^\beta - (m^\delta u_{3,}{}^\beta)|_\delta, \tag{42}$$

$$\begin{aligned} M^{\alpha\beta}|_{\alpha\beta} &+ (N^{\alpha\beta}u_{3,\beta})|_\alpha + (M^{\alpha\beta}u^\delta|_\beta)|_{\alpha\delta} - (M^{\alpha\delta}|_\alpha u^\beta|_\delta)|_\beta - [K^{\alpha\beta}(u_{3,}{}^\delta)|_\beta]|_{\alpha\delta} + p^3 + c^\alpha|_\alpha \\ &- (c^\alpha u^\beta|_\alpha)|_\beta + (c^3 u_{3,}{}^\alpha)|_\alpha = \not{f}^3 + m^\alpha|_\alpha - (m^\alpha u^\beta|_\alpha)|_\beta + (m^3 u_{3,}{}^\alpha)|_\alpha, \end{aligned} \tag{43}$$

after elimination of Q^α, T^α and N^{33} which takes place automatically when the variational principle is used. The rest of the field equations corresponding to this special case can be obtained in a similar fashion. Dropping terms involving $K^{\alpha\beta}$, m^i and c^3 in equations (42) and (43), we recover the equations of a classical non-linear plate theory due to Herrmann and Armenàkas [27].

That our results can be specialized to yield the classical von Kármán [28] non-linear plate equations has recently been illustrated by Ebcioglu [29].

Finally, complete linearization leads to

$$N^{\alpha\beta}|_\alpha + p^\beta = f^\beta, \tag{44}$$

$$\left.\begin{aligned} M^{\alpha\beta}|_\alpha - Q^\beta + c^\beta &= m^\beta, \\ Q^\alpha|_\alpha + p^3 &= f^3. \end{aligned}\right\} \tag{45}$$

Equations (45) are known to govern the small dynamic flexure of plates, keeping in mind that in the linear theory, as remarked by Eringen [15], the difference between the versions of Cauchy's laws expressed in the reference state and those in spatial form disappears.

5. CONCLUSION

The original version of our non-linear theory has so far been shown to include, as special cases, a classical non-linear theory of thin elastic plates as well as the more familiar von Kármán non-linear plate theory [29]. Further reduction for the non-linear analysis of thin, heterogeneous and anisotropic membranes subjected to mechanical and thermal loads is possible. The formulation being in tensor notation the results can readily be expressed in any particular coordinate system most suitable for the geometrical configuration at hand. Moreover, our two methods of derivation are not restricted to the special displacement assumption equation (15) adopted here for the main purpose of eventual correlation of our results with those known from earlier non-linear plate theories. In fact, except for this approximation, the results [equations (22)–(25), (27) and (37)–(39)] have been obtained through an exact derivation from three-dimensional continuum dynamics in terms of a reference state using the stress vector per unit area of the undeformed body. This latter point is of importance for the proper interpretation of various quantities that appear in non-linear theories of plates and shells.

Acknowledgements—The results presented here were obtained in the course of research sponsored by the National Aeronautics and Space Administration under Contract NAS8-5255 and by the National Science Foundation under Grant GP 515 and are part of a dissertation presented to the Graduate Council of the University of Florida in partial fulfillment of the requirements for the degree of doctor of philosophy. The author would like to thank Dr. I. K. Ebcioglu for his constant encouragement.

REFERENCES

[1a] C. TRUESDELL and R. TOUPIN, *Handb. Phys.* **3**, 553 (1960).
[1b] *Ibid.* **3**, 598 (1960).
[2] E. REISSNER, *J. Math. Phys.* **32**, 129 (1953).
[3] A. E. GREEN and J. E. ADKINS, *Large Elastic Deformations and Non-Linear Continuum Mechanics*. Clarendon Press, Oxford (1960).
[4] E. KOPPE, *Z. angew. Math. Mech.* **36**, 455 (1956).
[5] I. K. EBCIOGLU, *Int. J. Engng Sci.* **2**, 549 (1965).

[6] E. Hellinger, *Enz. math. Wiss.* **4**, 602 (1914).
[7a] T. C. Doyle and J. L. Ericksen, *Advances in Applied Mechanics* Vol. 4, p. 79 (H. L. Dryden and Th. von Kármán, Eds.) Academic Press, New York (1956).
[7b] *Ibid.*, p. 77.
[8] Hu, H.-C., *Sci. Sinica* **4**, 33 (1955) or *Acta Phys. Sinica* **10**, 259 (1954).
[9] K. Washizu, Technical Report 25-18, Section 2.4, Mass. Inst. of Tech., Aero Struct. Res. Lab. (1955).
[10] K. Washizu, Report 62-2, Dept. of Aero Eng., University of Washington, (1962).
[11] V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, p. 132. Graylock Press (1953); or *Theory of Elasticity*, p. 78. Israel Program for Scientific Translations (1961).
[12] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, Section 26. Pergamon Press, New York (1959).
[13] C. E. Pearson, *Theoretical Elasticity*, Chapter 10, Section 5. Harvard University Press (1959).
[14] W. Prager, *Introduction to Mechanics of Continua*, Chapter 10, Section 4. Ginn (1961).
[15] A. C. Eringen, *Nonlinear Theory of Continuous Media*, Art. 34. McGraw-Hill, New York (1962).
[16] T. Manacorda, *Boll. Un. Mat. Ital.* **9**, 154 (1954).
[17] E. Koppe, *Nachr. Akad. Wiss. Göttingen*,, **IIa**, 259 (1956).
[18] Yu, Y.-Y., *J. Acoust. Soc. Amer.* **36**, 111 (1964).
[19] A. E. Green and W. Zerna, *Theoretical Elasticity*, Section 5.4. Clarendon Press, Oxford (1954).
[20] P. M. Naghdi, *Progress in Solid Mechanics* Vol. 4, Chapter 1, Section 6, (I. N. Sneddon and R. Hill, Eds.). North-Holland (1963).
[21] R. F. S. Hearmon, *An Introduction to Applied Anisotropic Elasticity*, Section 5.4. Oxford University Press (1961).
[22] W. Nowacki, *Thermoelasticity*, Chapter 12, Section 1. Addison-Wesley (1962).
[23] J. L. Synge and A. Schild, *Tensor Calculus*, p. 69. University of Toronto Press (1949).
[24] K. Washizu, Report 1001, Mass. Inst. of Tech., Aero Struct. Res. Lab. (1960).
[25] B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*, Section 8.11. Wiley, New York (1960).
[26] A. S. Vol'mir, *NASA Tech. Transl. F-180* (1963); or *Raschet Prostr. Konstr.* **4**, 451 (1958).
[27] G. Herrmann and A. E. Armenàkas, *Proc. Amer. Soc. Civil Engrs. EM* (*J. Engng Mech. Div.*) **86**, 65 (1960).
[28] Th. von Kármán, *Enz. math. Wiss.* **4**, 311 (1910); or *Collected Works*, Vol. 1, pp. 141–207. Butterworth, London (1956).
[29] I. K. Ebcioglu, *Ing.-Arch.* **33**, 396 (1964).

(*Received* 1 *November* 1964)

Résumé—Une théorie non-linéaire de plaques est développée en se basant sur une théorie à trois dimensions d'élasticité en termes d'un état de référence. Ceci comprend les effets d'un cisaillement transversal et d'efforts normaux, de l'accélération, de l'anisotropie et de l'hétérogénéité du matériel et d'un champs de température. Les résultats s'appliquent aux plaques d'épaisseur constante et comprennent ceux de certaines théories précédentes non-linéaires en tant que cas spéciaux.

Zusammenfassung— Eine nicht-lineare Plattentheorie ist entwickelt auf der Grundlage von drei-dimensionaler Elastizitätstheorie mit Hilfe eines Referenzzustandes. Sie schliesst die Wirkung der Schiebung, normaler Verzerrung, Beschleunigung, Ungleichartigkeit des Materials und Anisotropie, und eines Temperaturfeldes ein. Die Ergebnisse gelten für Platten unveränderlicher Dicke und enthalten Ergebnisse von gewissen früheren nicht-linearen Plattentheorien als besondere Fälle.

Абстракт—Теория нелинейной пластины развилась на основе трёхмерной теории упругости в присудсцвии начального состояния. Зто включает эффект поперечного сдвига и нормальные деформации, ускорение, гетерогенность материала и анизотропию и поле температуры. Результаты годятся для пластин переменной толщины и содержат некоторые ранние теории нелинейных пластин, как особые случаи.